Exercise 38

If a wire with linear density $\rho(x, y, z)$ lies along a space curve C, its **moments of inertia** about the x-, y-, and z-axes are defined as

$$I_x = \int_C (y^2 + z^2)\rho(x, y, z)\, ds$$

$$I_y = \int_C (x^2 + z^2)\rho(x, y, z)\, ds$$

$$I_z = \int_C (x^2 + y^2)\rho(x, y, z)\, ds$$

Find the moments of inertia for the wire in Exercise 35.

Solution

The wire in Exercise 35 has a constant density k and is parameterized by $x = 2\sin t$, $y = 2\cos t$, $z = 3t$, where $0 \le t \le 2\pi$. Calculate I_x first.

$$\begin{aligned}
I_x &= \int_C (y^2 + z^2)\rho(x, y, z)\, ds \\
&= \int_0^{2\pi} \{[y(t)]^2 + [z(t)]^2\} k \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\
&= k\int_0^{2\pi} (4\cos^2 t + 9t^2)\sqrt{(2\cos t)^2 + (-2\sin t)^2 + (3)^2}\, dt \\
&= k\int_0^{2\pi} (4\cos^2 t + 9t^2)\sqrt{13}\, dt \\
&= \sqrt{13}k\left(4\int_0^{2\pi} \cos^2 t\, dt + 9\int_0^{2\pi} t^2\, dt\right) \\
&= \sqrt{13}k\left[4\int_0^{2\pi} \frac{1}{2}(1 + \cos 2t)\, dt + 9\left(\frac{t^3}{3}\right)\Big|_0^{2\pi}\right] \\
&= \sqrt{13}k\left[2\left(\int_0^{2\pi} dt + \int_0^{2\pi} \cos 2t\, dt\right) + 24\pi^3\right] \\
&= \sqrt{13}k\left[2\left(2\pi + \underbrace{\frac{1}{2}\sin 2t\Big|_0^{2\pi}}_{=0}\right) + 24\pi^3\right] \\
&= \sqrt{13}k(4\pi + 24\pi^3)
\end{aligned}$$

Therefore,

$$\boxed{I_x = 4\sqrt{13}\pi k(1 + 6\pi^2).}$$

Then calculate I_y.

$$\begin{aligned}
I_y &= \int_C (x^2+z^2)\rho(x,y,z)\,ds \\
&= \int_0^{2\pi} \{[x(t)]^2+[z(t)]^2\} k \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt \\
&= k\int_0^{2\pi} (4\sin^2 t + 9t^2)\sqrt{(2\cos t)^2+(-2\sin t)^2+(3)^2}\,dt \\
&= k\int_0^{2\pi} (4\sin^2 t + 9t^2)\sqrt{13}\,dt \\
&= \sqrt{13}k\left(4\int_0^{2\pi}\sin^2 t\,dt + 9\int_0^{2\pi} t^2\,dt\right) \\
&= \sqrt{13}k\left[4\int_0^{2\pi}\frac{1}{2}(1-\cos 2t)\,dt + 9\left(\frac{t^3}{3}\right)\Big|_0^{2\pi}\right] \\
&= \sqrt{13}k\left[2\left(\int_0^{2\pi} dt - \int_0^{2\pi}\cos 2t\,dt\right) + 24\pi^3\right] \\
&= \sqrt{13}k\left[2\left(2\pi - \underbrace{\frac{1}{2}\sin 2t\Big|_0^{2\pi}}_{=0}\right) + 24\pi^3\right] \\
&= \sqrt{13}k(4\pi + 24\pi^3)
\end{aligned}$$

Therefore,

$$\boxed{I_y = 4\sqrt{13}\pi k(1+6\pi^2).}$$

Then calculate I_z.

$$\begin{aligned}
I_z &= \int_C (x^2+y^2)\rho(x,y,z)\,ds \\
&= \int_0^{2\pi} \{[x(t)]^2+[y(t)]^2\} k \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt \\
&= k\int_0^{2\pi} (4\sin^2 t + 4\cos^2 t)\sqrt{(2\cos t)^2+(-2\sin t)^2+(3)^2}\,dt \\
&= k\int_0^{2\pi} (4)\sqrt{13}\,dt \\
&= 4\sqrt{13}k\left(\int_0^{2\pi} dt\right) \\
&= 4\sqrt{13}k(2\pi)
\end{aligned}$$

Therefore,

$$\boxed{I_z = 8\sqrt{13}\pi k.}$$